Quantitative and Symbolic Reasoning Center

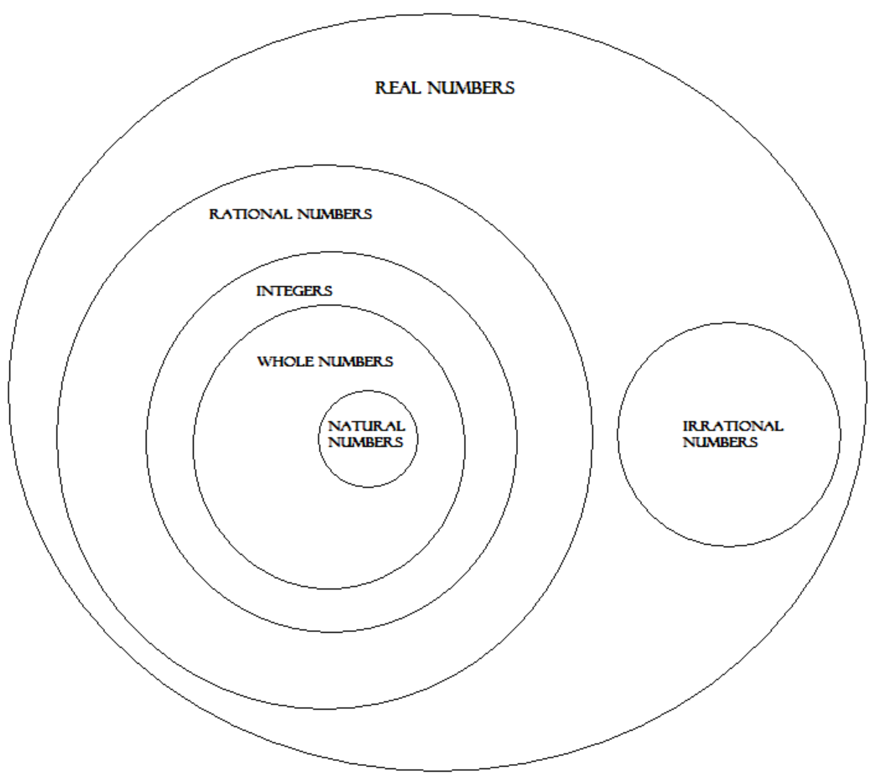
Hamilton College

Mathematical Review:
Algebra

Sets of Numbers

1. **Natural** numbers: Frequently called "counting numbers," they are also known as positive integers. (1, 2, 3, 4, …)
   Whole numbers: Comprise natural numbers and zero. (0, 1, 2, 3, 4, …)
2. **Integers**: Consists of positive whole numbers (0, 1, 2, 3, …) and negative whole numbers (-1, -2, -3, …).
3. **Rational** numbers: Consists of all numbers that can be written as the ratio of two integers (with a non-zero denominator) (1/4, -1/17, 11/3, 2/1), and whose decimal expansion is either terminating or repeating (2.0, 9.2300, 56.457457). All integers are rational numbers since they can be written as a ratio with a denominator of 1.
4. **Real** numbers: Comprises all numbers that can be written on a number line. All rational numbers are real numbers. However, not all real numbers are rational; some numbers are *irrational*. Irrational numbers are numbers that cannot be expressed as the ratio of two integers (π, √2, √3).

The Numerical Relationships:



It is often clear by inspection that a decimal representation of a number either terminates (ends in a string of zeros) or settles into a discernible pattern of repetition. It is less clear how to determine if a decimal representation *does not* have this behavior. Repetition may not occur until a place in the decimal representation far beyond that which is shown in writing. Thus, while the decimal representation of the number π, for example, is known to thousands of places, no regular pattern ever emerges!

Representing a terminating or repeating decimal number as a quotient of integers:

Ex. 1. 0.235 becomes $\frac{235}{1000}$

Ex. 2.

$N = 0.\overline{23} = 0.23232323232323...$ Multiply N by 100

$100N = 23.23 = 23.23232323...$ Subtract the two equations

$99N = 23$

$N = \frac{23}{99}$

Practice:

I. Classify the numbers below with as many of these names as you can:
"natural number," "whole number," "integer," "rational number," "irrational number," and "real number."

a. 12	c. 6/2	e. $\sqrt{4}$	g. 8.14	i. 0
b. -7	d. $\sqrt{10}$	f. π	h. 0.333	j. -11/9

II. Write the appropriate symbol (= or ~) between each pair of numbers.

a. 1/2 ______ 12/24	d. 3/4 ______ 0.75
b. 6/7 ______ 0.8571428571426	e. 18 ______ 36/2
c. π ______ 3.14	f. $\sqrt{11}$ ______ 3.3166

III. Tony claims that π is equal to 22/7. He argues that since π is an irrational number, and irrational numbers cannot be expressed as repeating decimals, that 22/7 cannot be expressed as a repeating decimal.

Do you agree with Tony? Justify your reasoning.

IV. Express the following decimals as a quotient of integers.

a. 0.1 b. 0.2 c. 0.01 d. $0.15\overline{47}$

Ratios and Proportions

A **ratio** is a fractional representation of two measured quantities. (Hence the term RATIOnal number!)

Ex. The ratio of 200 minutes to two hours is: $200/(2\cdot 60) = 5/3$

A **proportion** is a statement that equates two ratios.

Ex. Sarah is driving home for spring break. In her Saturn Ion she can travel 400 miles on 20 gallons of gas. At this rate, how many more gallons will she need for a 900 mile trip?

Let G represent the number of gallons used. The proportion would look like:

$$\frac{20\text{ gallons}}{G\text{ gallons}} = \frac{400\text{ miles}}{900\text{ miles}}$$

$G = (20g\cdot 900mi) = (Gg\cdot 400mi) \rightarrow G = 45$ gallons

Part-whole Percentage Problems

The basic equation for determining a percentage or part is:

$$\frac{\text{Part}}{\text{Whole}} = \frac{\text{Percentage}}{100}$$

Ex. 1 Daniel purchases a Hamilton sweatshirt from the bookstore. The sweatshirt is regularly \$50, but is on sale for 40% off the regular price. How much does Daniel pay?

$$\frac{x\text{ (unknown part)}}{50\text{ (whole)}} = \frac{40\text{ (percentage)}}{100}$$

$$100x = 2000$$

$$x = 20\text{ dollars}$$

Thus, 40% of 50 is 20. To determine how much Daniel pays for the sweatshirt, we need to subtract the 40% off (20 dollars) from the regular price. So, Daniel pays 30 dollars.

Ex. 2 Commons orders an extra box of oranges by mistake. By the end of the week there are 70 oranges remaining and 21 of them are rotten. What percentage of the oranges is rotten?

$$\frac{21\text{ (part)}}{70\text{ (whole)}} = \frac{p\text{ (percentage)}}{100}$$

$$2100 = 70p$$

$$p = 30$$

Ex. 3 Professor Davenport earns a salary of \$60,000. Last year she received a 25% raise. This year, however, she took a 25% cut. What was her salary last year? This year?

a. $\frac{x}{60{,}000} = \frac{25}{100} \rightarrow 100x = 1{,}500{,}000 \rightarrow 15{,}000 \rightarrow 60{,}000 + 15{,}000 = 75{,}000$

b. $\frac{x}{75{,}000} = \frac{-25}{100} \rightarrow 100x = -1{,}875{,}000 \rightarrow -18{,}750 \rightarrow 75{,}000 + (-18{,}750) = 56{,}250$

Practice:

I. The Red Sox played against the Yankees in eighteen different matches before the finals. If the Red Sox won twelve of those matches,

a. What is the ratio of the number of games won by the Red Sox to the number of games played?

b. What is the ratio of the number of games lost by the Sox to the number of games played?

c. What is the ratio of the number of games won by the Sox to the number of games lost?

II. In one section of Stats, the ratio of males to females is three to two.

a. What is the ratio of females to males?
b. If there are 20 students in the class, how many are female?
c. If there are 24 females, how many students are there?
d. Explain why there cannot be 16 students in the class.

III. If a 6 foot tall man casts a 10 foot shadow, and the tree he is standing next to casts a 24 foot shadow, how tall is the tree? (Hint: draw a picture. Recall similar triangles from geometry? This problem can be solved both algebraically and geometrically.)

IV. The co-op has recently expanded their 1812 garden by taking over the soccer and football fields. If they decide to plant 2 acres of carrots, and they can plant 1/5 of an acre a day, after three days what is the percentage of acres planted?

V. The latest 32 GB iPhone is on sale for $90. If the salesman says this price is only 30% of the original price, what was the original price of the phone?

VI. The Kirkland Art Center had its annual art sale and made a profit of $7,000. If the center made 12% more last year, how much did they make?

Solving Algebraic Expressions:

When **solving an equation for a variable**, we are trying to find the value of the variable. Sometimes we can get an exact numerical value for the variable, and sometimes we can just get the value of the variable as an **algebraic expression** of the other variables in the equation. Generally, this variable is x, and we create a sequence of simpler equivalent equations* until only the variable x is on one side of the equation.

*Two equations are **equivalent** if one equation is obtained from the other by:

a. *Adding* the same number to each side
b. *Subtracting* the same number from each side
c. *Multiplying* both sides by the same non-zero number
d. *Dividing* each side by the same non-zero number

Ex. 1. Solve $5x = 45$.

Solution

Step 1: $\frac{5x}{5} = \frac{45}{5}$ *(divide each side by 5)*

Step 2: $x = 9$ *(cancel out superfluous numbers and rewrite the equation)*

Ex. 2. Solve $\frac{x}{4} + 8 = 23$.

Solution

Step 1: $\frac{x}{4} + 8 - 8 = 23 - 8$ *(subtract 8 from each side)*

Step 2: $\frac{x}{4} * 4 = 15 * 4$ *(multiply each side by 4)*

Step 3: $x = 60$ *(cancel and simplify)*

The distributive property states that $a*(b \pm c) = ab \pm ac$. Because multiplication is commutative, order does not matter, $(b \pm c)*a = ba \pm ca$. The distributive property is most often used to combine like terms and remove parenthesis.

Ex. 3. Simplify $4x + 7y + 9x - 3$.

Solution

$4x + 7y + 9x - 3 = (4x + 9x) + 7y - 3 = 13x + 7y - 3$

Ex. 4. Simplify $6(2 - 5x)$.

Solution

$6(2 - 5x) = 6(2) - 6(5x) = 12 - 30x$

Ex. 5. Solve $5(3x + 12) = 14x - 9x$.

Solution

Step 1: $15x + 60 = 14x - 9x$	*(distribute)*
Step 2: $15x + 60 = 3x$	*(add like terms)*
Step 3: $12x + 60 = 0$	*(subtract 3x from each side)*
Step 4: $12x = - 60$	*(subtract 60 from each side)*
Step 5: $x = -5$	*(divide each side by 12)*

An equation stating that two equations are equal is called a *proportion*. To solve this type of equation, you use the method called cross multiplication.

$$\frac{a}{b} = \frac{c}{d} \rightarrow ad = bc$$

Ex. 6. Solve $\frac{18}{x} = \frac{32}{48}$

Solution

Step 1: $18(48) = 32(x)$	*(cross multiply)*
Step 2: $864 = 32x$	*(simplify)*
Step 3: $27 = x$	*(divide each side by 32)*

Practice:
Solve the equation for x.

a. $4x + 18 = 54$ b. $5x + 17x = 7(12 - 3x)$ c. $45 - x = 2x$

d. $\frac{12}{16} = \frac{x}{8}$ e. $\frac{150}{x-4} = \frac{225}{x+5}$ f. $3 = \frac{2x + 1}{x - 1}$

Writing Algebraic Expressions:

Solving word problems can be tricky. Mathematically, you need to find an equation that models the situation presented in the word problem, and then solve this equation for a particular variable. The most difficult part of this is coming up with an appropriate equation. Below are some tips for modeling and solving word problems.

a. Determine what information is given in the problem, and what you need to find. Assign a letter or symbol (i.e. a variable) to each quantity that plays a role in the problem.
b. Organize your information—draw a picture or a graph if it helps you to understand the problem better. Look for a relationship between the variables you identified in the previous step, using given information, geometry or trigonometry. Express this relationship as an equation.
c. Use the information in the problem to replace as many variables as possible with known quantities, until you are left only with the letter representing the unknown quantity you need to find. Then solve the equation for that variable.
d. Check your answer for the appropriate units, and that the answer makes sense in the context of the equation.

Ex. 1. If Sarah has three nickels and four dimes, how much money does she have?

$3(N) + 4(D) = 3(5) + 4(10) = 55$ cents

Ex. 2. For Late Nite catering, Bon Appetite charges $4.75 per person, but after the first fifty people, Bon Appetite charges $2.15 a person. If 183 people show up to the Knit Happens Late Nite event, how much will the club be charged?

Solution
Start by explicitly identifying the important variables. In this case, let x represent the amount the club is charged, and let P represent the total number of people at the event.

$x = 4.75(50) + 2.15(P-50)$
$x = 4.75(50) + 2.15(183-50)$ *(substitute in known quantities)*
$x = 4.75(50) + 2.15(133)$ *(simplify and solve)*
$x = 237.50 + 285.90 = \$523.40$

Practice:
Write the algebraic expression or equality.

I.a. The value, in cents, of D dimes and Q quarters

b. 48 minus 7 times a number

c. Twelve more than a number is six times the number

d. The number of points scored by the basketball team in a game in which they made *f* foul shots (one point each), twenty-three two-point shots, and *t* three-point shots.

e. The sum of two numbers is thirteen more than their product.

Solve

II.f. The Little Pub found that 1/5 of all their pint glasses are "misplaced" by students by the end of the year. If 50 glasses are missing, how many pint glasses did the Pub have?

g. A waiter at Zebb's makes $8.25 an hour, and time-and-a-half for each hour beyond 40 hours a week. If Leroy's gross earnings (before taxes) was $404.25, how many hours did he work that week?

h. Campus police issued eighteen more parking tickets in the month of May then they did in April. The total number of tickets over those two months was 128. How many tickets were issued in April? (Let t represent tickets in April.)

i. Journey will be giving a concert at Hamilton. The tickets prices are $15 for students and $20 for non-students. If 1,637 students bought tickets, and the total revenue is $32,695, how many non-students bought tickets?

An equation involving two variables—say x and y—expresses a relationship between two quantities. If we can solve this equation for y, then we end up with an equation in the form:

y = some algebraic expression involving x.

For any given value of the variable x, we can use this equation to find the value of y. In mathematical terminology, we say y is a **function** of x, and write $y = f(x)$. Since the value of y **depends** on the value of x, we call y the **dependent** variable, and x the **independent** variable. The graph of a function is the set of all points in the Cartesian (x-y) plane in which the y-coordinate can be calculated from x-coordinate using the algebraic expression involving x. Certain families of functions have been found very useful in modeling real-world problems.

Linear Functions

Linear functions are the simplest possible functions (this is true of ALL equations between two variables, linear and non-linear). A natural example is the relationship between the Fahrenheit and Celsius temperature scales:
$F = 9/5C + 32$, where F represents degrees Fahrenheit and C represents degrees Celsius.

Ex.1. If the temperature is 68 degrees Fahrenheit, what is it in Celsius?

$$68 = (9/5)C + 32$$
$$36 = (9/5)C$$
$$C = (5/9)(36) = 20$$
$$C = 20$$

In general, a linear function has the form $y = mx + b$. The graph of a linear function is a straight line. This equation is called the **slope-intercept form**, where m represents the slope (a measure of the steepness of the line; measuring the change in the y-value per unit change in the x-value) and b is a constant number (when graphed, b is the y-intercept). In the temperature example, the equation $F = (9/5)C + 32$ is expressed in slope-intercept form with $m = 9/5$ and $b = 32$.

If two points on a line are known, we can obtain the equation, in slope-intercept form, of the line containing the two points. (Strictly speaking, however, this fact is only true of non-vertical lines.) Using our temperature example, we know that 0° Celsius corresponds to 32° Fahrenheit, and 100° Celsius corresponds to 212° Fahrenheit. Thus, the points (0, 32) and (100, 212) are on the line that graphs the relationship between the Fahrenheit and Celsius scales.

Using the slope-intercept form:

$$m = \frac{\Delta\ y\text{-value}}{\Delta\ x\text{-value}} = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{9}{5}$$

So, the relationship is $F = (9/5)C + b$, where b is yet unknown. However, because we know that 0° C corresponds to 32° F, we can substitute those numbers into the equation:

$$32 = (9/5)(0) + b$$
$$32 = b.$$

Thus, the equation is $F = 9/5C + 32$.

Practice:

I. Hamilton Zip-cars charges 25 dollars a day and 10 cents per mile.

a. What is the charge for renting a car for one day and driving 100 miles?
b. What is the charge for renting a car for two day and driving 250 miles?
c. If Shari rents a car for one day and was charged $75, how many miles did she drive?

II. Calculate the slope of the line.

a. (1,1) and (2,4)
b. (-2,-4/5) and (1/2, 2/3)
c. (a,b) and (b,a) where $a \neq b$

III. Write an equation in slope-intercept form expressing y as a linear function of x.

a. When $x = 5$, $y = 0$, and when $x = 10$, $y = 1$
b. When $x = 15$, $y = -1$ and when $x = 1$, $y = 1$
c. When $x = 0$, and $y = 0$ and when $x = 1$, $y = 1$
d. When $x = ½$, $y = -2$, and when $x = -3$, $y = 1/3$

IV. First solve each equation for y. Then find the slope and y-intercept of each of the following lines.

a. $x – y = 1$
b. $x/2 – 2y = -2$
c. $x = (y – 1) / 3$

Exponential Functions

One of the most important types of non-linear functions is **exponential functions**. They are used to model situations with a very special property; the **rate of change** in y is proportional to the **value** of x. These functions have the general form of $y = ca^x$ where a is the growth factor and c is the y-intercept. When the growth factor is > 1 we have *exponential growth*; when the growth factor is $0 < a < 1$ we have *exponential decay*.

Exponential functions are often used to model population growth, interest rates on savings accounts, and exponential decay of a radioactive element.

When determining exponential models of growth, there is a simple equation:

$$P(t) = P_0 e^{kt}$$

where P_0 represents initial population, k represents **relative growth rate**, and t represents time. Relative growth rate can be determined by the formula: $k = (1/P(t))(dP/dt)$, where P(t) is population at time t, and dP/dt represents change in population / change in time.

Ex. 1. Repeating G. F. Gause's experiment with the protozoan *Paramecium*, Damon's Bio class wants to determine the exponential model of growth. They discovered that for an initial population of 2, the initial relative growth rate was 0.7944.

Solution

$$P(t) = P_0 e^{kt} = 2(e^{.7944t}) = 2e^{.7944t}$$

Ex. 2. The half-life of radium-226 is 1590 years. A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after *t* years.

Solution
We want to determine the exponential model of growth for radium-226. So, let $m(t)$ be the mass of radium-226 that remains after *t* years. Then $dm/dt = km$ and $m_0 = 100$. So, we have:

$$m(t) = m_0 e^{kt} = 100e^{kt}$$

To determine k:
We know that after 1590 years (t) the 100 mg (m_0) will reduce to 50 mg. So,

$100e^{1590k} = 50$

$e^{1590k} = ½$

$1590k = \ln ½$ *(introduce natural log on both sides)*

$1590k = - \ln 2$ *(natural log rule: ln ½ = ln 1 – ln 2)*

$k = -\frac{\ln 2}{1590}$

Thus, $m(t) = 100e^{-(\ln 2)t / 1590}$
Simplified: $m(t) = 100 \times 2^{-t / 1590}$

Practice:

I. For her senior thesis, Louise created a new type of bacteria. If a cell of *louisanus toxicus* divides into two cells every 20 minutes, and on day zero she began with two cells:

a. Find the relative growth rate.
b. Find an expression for the number of cells after t hours.
c. Find the number of cells after 8 hours.

II. Bismuth-210 has a half-life of 5.0 days. If the original sample is 800 mg, (a) find a formula for the mass remaining after t days, and (b) find the mass remaining after 30 days.

II. Exponential functions are an easy way of determining interest on a savings account. Generally, for an amount A_0 invested at an interest rate r, after t years the account is worth $A_0(1 + r)^t$. However, interest is usually compounded more frequently then just at the end of the year. So, for n times a year the interest rate is r/n and there are nt compounding periods in t years:

$$A_0(1 + {}^r/_n)^{nt}$$

For instance, if $3,000 is invested at 2% interest for three years it will be worth:

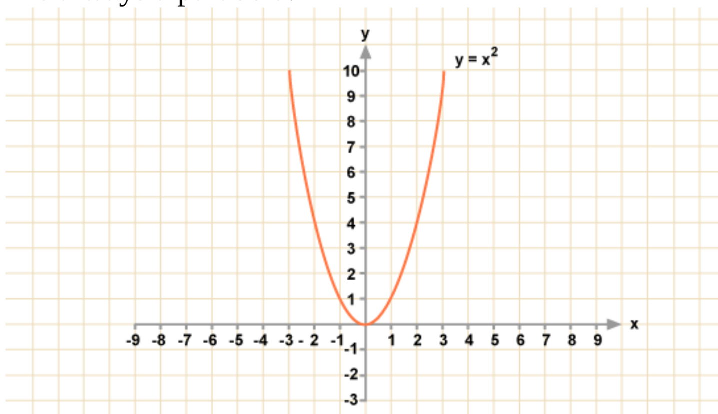
$3000(1.02)^3 = $3,183.62 *(annual compounding)*
$3000(1.002)^36 = $3,223.73 *(monthly compounding)*

After winning a fist pump competition on the latest MTV reality show, Jake invests his $5,000 into a savings account. If Jake invests his money for 2 years at 3%, what will his investment be worth with

a. annual compounding?
b. semiannual compounding?
c. quarterly compounding?
d. monthly compounding?
e. daily compounding?

Quadratic Functions

A **quadratic function** is one of the form $y = ax^2 + bx + c$, for $a \neq 0$; the graph of a quadratic function is always a parabola.



The y-intercept is found by substituting x = 0 into the equation. For this function, that means the y-intercept is *c*. There may be one, two, or no x-intercepts. The x-intercepts are found by substituting y = 0 into the equation and then using the **quadratic formula**.

Quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Ex. 1. $y = x^2 - 6x + 2$. What are the x- and y-intercepts?

y-intercept: $y = (0)^2 - 6(0) + 2 \rightarrow y = 2$

x-intercept: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(2)}}{2(1)}$

$x = \frac{6 \pm \sqrt{28}}{2}$

$x = \frac{6 \pm 5.292}{2}$ = x = 5.646 and 0.354

Practice:

Find the x- and y-intercepts.

I. $y = 4x^2$ II. $y = x^2 + 4x + 2$ III. $y = 3x^2 - 7x - 1$

Averages/Means

An **arithmetic mean**, commonly called an **average**, is a method used to determine the central tendency for a sample space. In simplest terms, it is the sum of all scores divided by the number of scores.

Ex. 1. Tiffany received scores of 81, 90, 84, and 89 on her Organic Chemistry tests. Her average is: $\frac{81 + 90 + 84 + 89}{4} = 86$

Ex. 2. For Calculus, homework counts as 15% of the final grade, the exams as 20% each, and the final exam as 25%. If Toby has a 94 homework average, received 91, 92, and 95 on the three exams and a 95 on the final, what is his final grade?

Solution

94(.15) + 91(.20) + 92(.20) + 95(.20) + 95 (.25) =
14.1 + 18.2 +18.4 + 19 + 23.75 = 93.45 = 93

Practice:

a. Ryan received scores of 92, 86, and 83 on his first three physics tests. What score does he need on the fourth test for this test average to be 90?

b. If Deshnae received a 92 as her final average in Calc. II, and her exam grades were 89, 93, and 91, and her final exam grade was 86, what was her homework average?

c. Ralph's scores on his five bio tests improved by two points each time. His average on his five tests was 90. What were Ralph's scores on his five tests?

d. The Colgate basketball team averaged 84.8 points per game; the Continental's averaged 91.2 points per game. A Colgate player bragged that although the Continental's averaged more points per game, Colgate actually scored more total points over the course of the season. Could this be possible?

In certain situations it is convenient to know the average speed you were driving, or how long a trip took you. There is a simple equation for determining **average velocity**:

$$\text{Average Velocity} = \frac{\text{change in distance}}{\text{change in time}}$$

Ex. 3. Frederick is driving home for break. After one hour he has traveled 45 miles. In the second hour, he drove 65 miles. What is Frederick's average velocity?

Solution

The total distance covered was 110 miles in 2 hours. Thus, 110 miles / 2 hours equals 55 mph.

Practice:

e. Jing drives to her friend’s house at an average velocity of 52 miles per hour. If the distance is 138 miles, how long will it take her to get there?

f. Luke enters the New York State Thruway in Utica and drives 45 miles to Syracuse, where, upon exiting the highway, he is rebuked by the toll booth attendant for having exceeded the 55 mile per hour speed limit at some point in his trip. How could the toll booth attendant be sure of this?

g. Marika jogs once around the ¼ mile track at a rate of 4.5 miles per hour, and runs once around the track at a rate of 6.5 miles an hour. What is her average velocity?